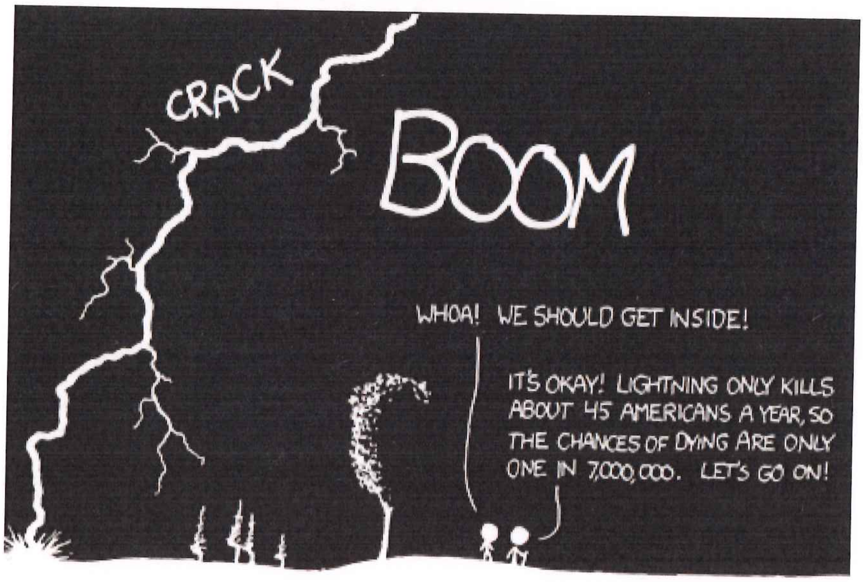
3.13 Probability



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

Ms Walker's

Introduction

This standard will require you to applying probability concepts in solving problems. This will involve the use of:

- true probability versus model estimates versus experimental estimates
- randomness
- independence
- mutually exclusive events
- conditional probabilities
- probability distribution tables and graphs
- two way tables
- probability trees
- Venn diagrams

What is Probability?

"I wish we hadn't learned probability 'cause I don't think our odds are good."

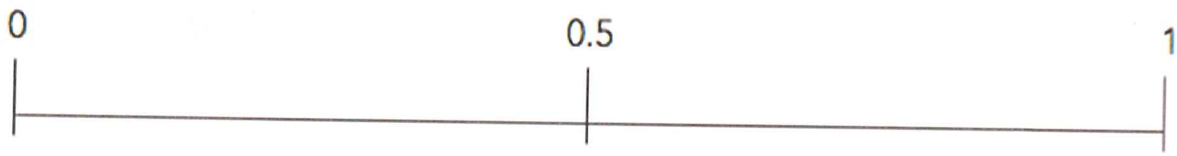
These are the formulae that will supplied to you in the external examination.

Permutations and combinations	$^{n}P_{r} = \frac{n!}{(n-r)!}$ $\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{(n-r)!r!}$
Probability	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Probability Revision

Probabilities can be expressed as a decimal, fraction or a percentage.

We usually use decimals because they are easy to compare.



Ways of calculating probabilities:

1 Equally likely outcomes: Probability = $\frac{\text{number of favourable outcomes}}{\text{total possible outcomes}}$

Example: For a 52-card pack, P(ace) = $\frac{4}{52}$

= ____________

2 Long run relative frequency: Probability = $\frac{\text{number of times an event occurs}}{\text{total number of trials}}$

Example:

P(Mike bikes to school) = $\frac{\text{number of times he has biked to school in the last year}}{\text{total number of school days in the last year}}$

Combining probabilities

This is very important and frequently comes up in exams.

and ____x____

or ____+____

Multiplication Principle

1. With different items:

eg Jeremiah has 6 shirts, 2 ties and 4 pairs of trousers
In how many ways can he be dressed?

$6 \times 2 \times 4 = 48$

2. Where items are similar (and all items are available for positions) but no repeats are allowed:

eg 10 people are running a race - in how many ways can the first 3 places be filled.

$10 \times 9 \times 8 = 720$

3. Where items are similar, but repeats are allowed.

eg number of New Zealand license plates

$25 \times 25 \times 25 \times 10 \times 10 \times 10 = 156\,250\,00$

Factorials

Howmany different ways can 4 people sit on a bench?

$4 \times 3 \times 2 \times 1 = 24$

Howmany different ways can 12 people sit on a bench?

$12! = 479\,001\,600$

Challenge: How many different ways can 4 people sit around a circular table?

$$\frac{4 \times 3 \times 2 \times 1}{4} = 3!$$

$$= 6$$

Pg15-16

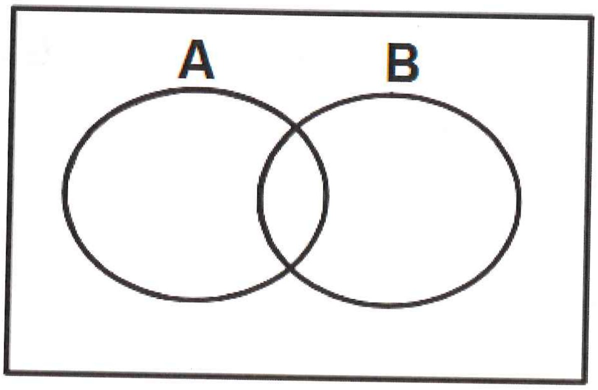
Probability Notation

$P(A)$	The probability of A occurring
$P(A')$	The probability of A not occurring
$P(A \cap B)$	The probability of A **and** B occurring
$P(A \cap B)'$	The probability of A **and** B not occurring
$P(A \cap B')$	The probability of A occurring **and** B not occurring
$P(A \cup B)$	The probability of A **or** B occurring
$P(A' \cup B)$	The probability of A not occurring **or** B occurring
$P(A \cup B)'$ or $P(A' \cap B')$	The probability of A or B not occurring
	So **neither** occurring

Questions for 2x2 Venn or Two way contingency tables

The data may be given in fractions, decimals, percentages or frequencies (ie numbers of).

Example 1: Draw a Venn diagram and a probability table to show the following probabilities: P(A) = 0.5, P(B) = 0.6 and P(A ∩ B) = 0.2. Use them to find P(A ∪ B).



	A	A'	
B	0.2	0.4	0.6
B'	0.3	0.1	0.4
	0.5	0.5	1

Check that all the probabilities add to 1

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

On your formula sheet

Find P(A ∪ B) = $0.2 + 0.4 + 0.3$

$= 0.9$

→ or $0.5 + 0.6 - 0.1$

$= 0.9$

(iii) A probability model has been developed for flights departing from another airport.

Let A be the event "a flight's departure time is affected by passenger behaviour".

Let B be the event "a flight's departure time is affected by weather conditions".

Under this model, $P(A \cup B) = 0.54$ and $P(A' \cup B) = 0.86$.

What is the probability that a flight's departure time is affected by weather conditions? ie $P(B)$

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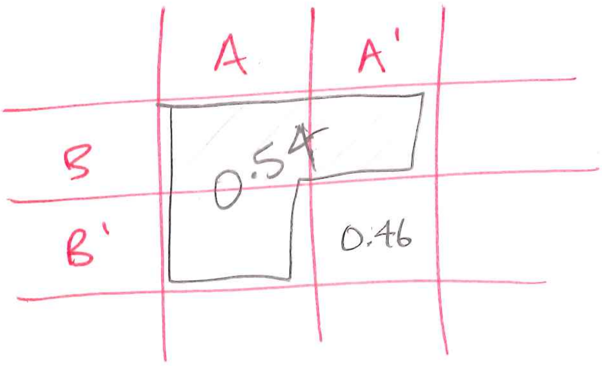
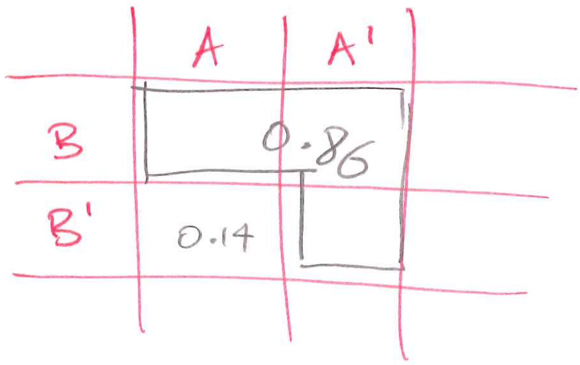




$$P(B') = 0.46 + 0.14$$
$$= 0.6$$

$$\therefore \quad P(B) = 1 - 0.6$$
$$= 0.4$$

3x3 Frequency table

Example: A group of 215 Year 13 students were asked which of the subjects Art, Biology and Chemistry they were studying.

- 3 students studied all three subjects.
- 11 students studied Art and Biology.
- 26 students studied Biology and Chemistry.
- 5 students studied Chemistry and Art.
- 50 students studied Art.
- 80 students studied Biology.
- 60 students studied Chemistry.

Draw a Venn diagram to represent this situation and use it to calculate the number of students who take none of these subjects.

<table>
<tr><td></td><td></td><td>C</td><td>C'</td><td></td></tr>
<tr><td rowspan="2">A</td><td>B</td><td>3 5</td><td>11 8</td><td rowspan="2">50</td></tr>
<tr><td>B'</td><td>2</td><td>37</td></tr>
<tr><td rowspan="2">A'</td><td>B</td><td>26-3 23</td><td>80-23-8-3 46</td><td rowspan="2">165</td></tr>
<tr><td>B'</td><td>32</td><td>64</td></tr>
<tr><td colspan="2"></td><td>60</td><td>155</td><td>215</td></tr>
</table>

64 took none.

pg31-35

What's wrong with this one?

From the tour company's records for a randomly selected day, the following information was found:

- 22 passengers were male
- 10 passengers were Australian
- 8 passengers were aged over 30
- 1 passenger was a male Australian aged over 30
- 3 passengers were male Australians aged 30 or younger
- there were no female Australians aged 30 or younger
- 2 passengers were males aged over 30, but were not Australian
- there were 30 passengers altogether.

Find the probability that a randomly selected passenger for that day was a female Australian aged over 30.

		M	F	
A	<30	3	0	10
	30+	1	6	
A'	<30	16		
	30+	2		
		22		30

Passengers over 30 already adds to 9.

Tree Diagrams

These can be used as an alternative to venn diagrams and tables, they are particularly useful when there is a sequence of events.

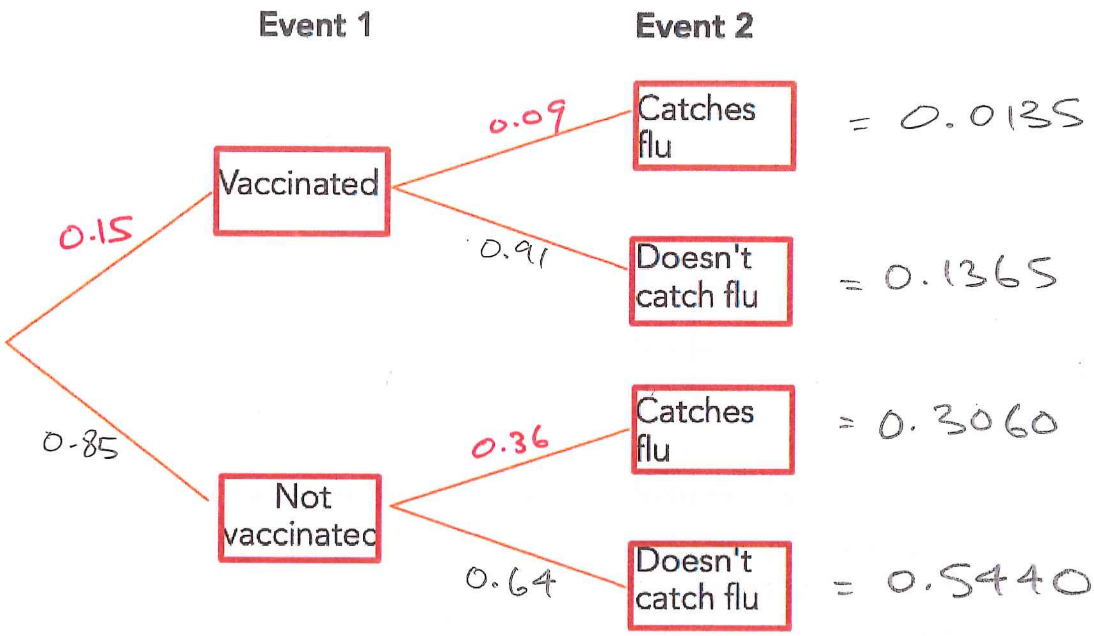
Example: In a particular region, 15% of the population is vaccinated against the flu. The probability that a vaccinated person catches the flu is 0.09, but 36% of unvaccinated people will catch the flu.

1. First identify the events are and which order they go in

   vaccinated and flu

   Which is likely to happen first? vaccinated

2. Add the events and the probabilities to the tree diagram below



Now answer the questions

a Calculate the probability that a vaccinated person catches the flu.

$P(V \cap F) = 0.0135$

b Calculate the overall percentage of people who will catch the flu.

$0.0135 + 0.3060 = 0.3195 \quad \therefore \ 31.95\%$

c What percentage of those who catch the flu had been vaccinated?

out of those who catch $\frac{0.0135}{0.3195} = 0.04225 \quad \therefore \ 4.23\%$

d What percentage of vaccinated people caught the flu?

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out of vaccinated people $\frac{0.0135}{0.15} = 0.09 \quad \therefore \ 9\%$

Question Three 2017

(a) Data was obtained on all flights that departed from Wellington Airport during one day in January 2017.

For the 83 flights that had departure time data available:

- 64 flights were operated by Air New Zealand
- 31 flights were delayed
- 12 flights were not operated by Air New Zealand and were not delayed.

(i) Suppose one of these flights is chosen at random.

Calculate the probability that this flight was delayed, given that the flight was not operated by Air New Zealand.

	D	D'	
ANZ			64
ANZ'		12	
	31		83

$\frac{7}{19} = 0.368$

Question Four

A sample of 996 students in Years 9 to 13 was taken from the Census at School 2015 database.

(a) 78.4% of these students were born in New Zealand.

2017

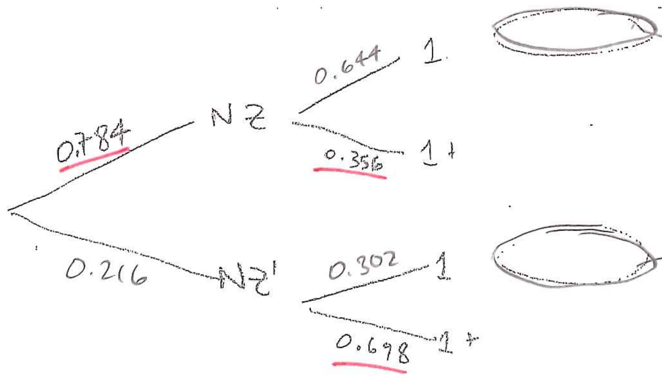
100% of these students can speak at least one language fluently.

Of the students born in New Zealand, 35.6% can speak more than one language fluently.

Of the students not born in New Zealand, 69.8% can speak more than one language fluently.

A student from the sample is chosen at random.

(i) Calculate the probability that the student can speak only one language fluently.



$(0.784 \times 0.644) + (0.216 \times 0.302) = 0.5701$

Question Seven

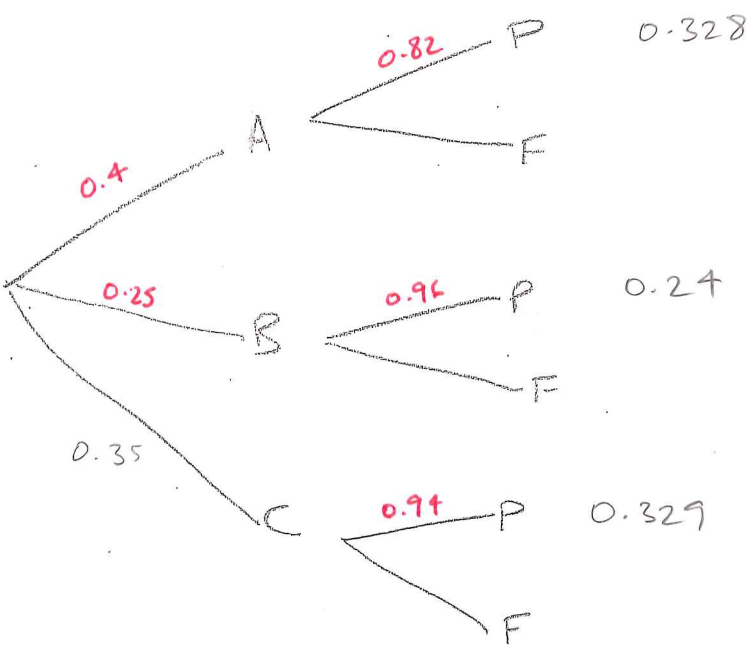
(a) People take their cars to testing centres for a Warrant of Fitness (WOF).

Three testing centres were recently reviewed over a one-month period: testing centre A, testing centre B, and testing centre C. During this time, all results for tests completed by each of the testing centres were recorded.

40% of the tests reviewed were completed by testing centre A, and 25% of the tests reviewed were completed by testing centre B.

Of the tests completed by testing centre A, 82% were successful (the car passed the WOF).
Of the tests completed by testing centre B, 96% were successful.
Of the tests completed by testing centre C, 94% were successful.

(i) What percentage of tests completed during the review were successful?

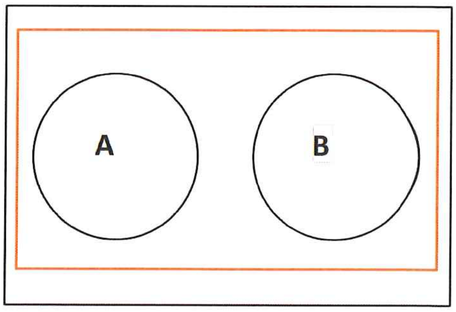


$0.328 + 0.24 + 0.329 = 0.897$

Mutually Exclusive Events

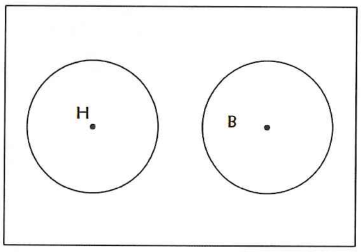
Mutually exclusive events cannot both occur. In a Venn Diagram the circles don't overlap.

$P(A \cup B) = P(A) + P(B)$



Example: Hearts and black cards are mutually exclusive.

Find the probability of picking a heart or a black card.



$P(H) + P(B) = \frac{1}{4} + \frac{1}{2}$

$= 0.75$

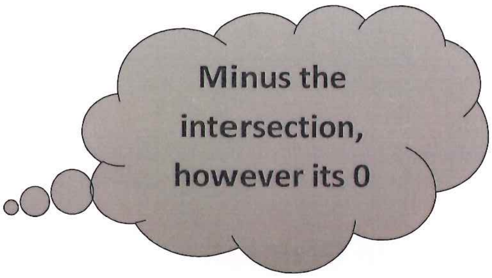
	B	B'	
H	0	.25	.25
H'	.5	.25	.75
	.5	.5	1.0

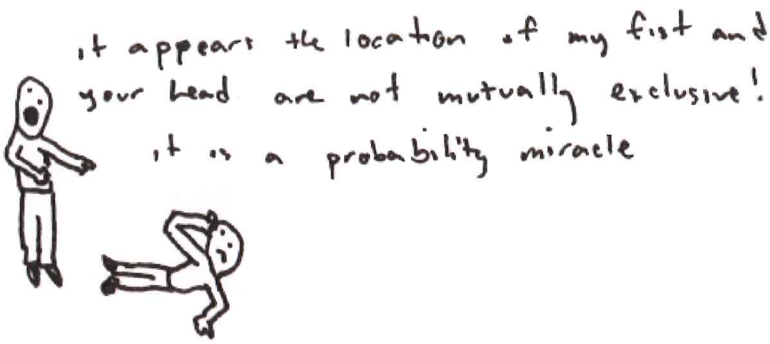
In General:

$$P(A \cup B) = P(A) + P(B)$$

and

$$P(A \cap B) = \emptyset$$





Question One

On a particular day, P(rain) = 0.27, P(heavy wind) = 0.24, P(neither) = 0.64.

Are these events mutually exclusive?

	R	R'	
W	0.15		0.24
W'	0.12	0.64	0.76
	0.27		1

Is this 0?

No, so not mutually exclusive.

Question Two

2018

(a) A 2017 food marketing study from New Zealand examined 70 websites belonging to the most popular food and drink brands. 24 of these websites were targeted at teenagers, while the others were targeted at the general population. 21 of the websites made a positive health claim, and of these websites, eight were targeted at teenagers.

One of the websites is chosen at random.

(i) Explain why the events "a website makes a positive health claim" and "a website is targeted at the general population" are not mutually exclusive.

Support your answer with at least one calculation.

	T	T'	
+H	8	13	21
-H'			
	24		70

$P(+H \cap T') = \frac{13}{70}$

which is not 0

$\therefore$ not mutually exclusive

Question Three

2015

In 2013, 63.9% of imported cars registered with the New Zealand Transport Agency were manufactured in Japan. Of these cars manufactured in Japan, 80.3% were used cars.

Suppose that one of the imported cars registered with the New Zealand Transport Agency in 2013 was selected at random.

(i) Explain why the events "The car was manufactured in Japan" and "The car is a used car" are not mutually exclusive.

Include statistical reasoning in your explanation.

0.639 J — 0.803 U — 0.513117; N

J'

	Jap	Jap'	
Used	51.317		
New			
	63.9		100

63.9 x 0.803

Because 51.3% are Jap & used they are not mutually exclusive

Example: Consider the subject choices of some Year 13 students.

a The probability that a student takes Art is 0.15, and the probability that a student takes both Art and Design is $0.08\dot{3}$. Calculate the probability that a student takes Design, given that he or she takes Art.

Step 1: Write in symbols what you know. $P(A) = 0.15$ and $P(D \cap A) = 0.08\dot{3}$

Step 2: Write down what you need to know. $P(D/A)$

Step 3: Use the formula.

$$P(D/A) = \frac{P(D \cap A)}{P(A)}$$

Use appropriate letters.

$$= \frac{0.08\dot{3}}{0.15}$$

$$= 0.\dot{5}$$

Alternative method: Use reasoning and a probability table (you have been doing this in previous sections of the book).

	A	**A'**	
D	**$0.08\dot{3}$**		
D'			
	0.15		

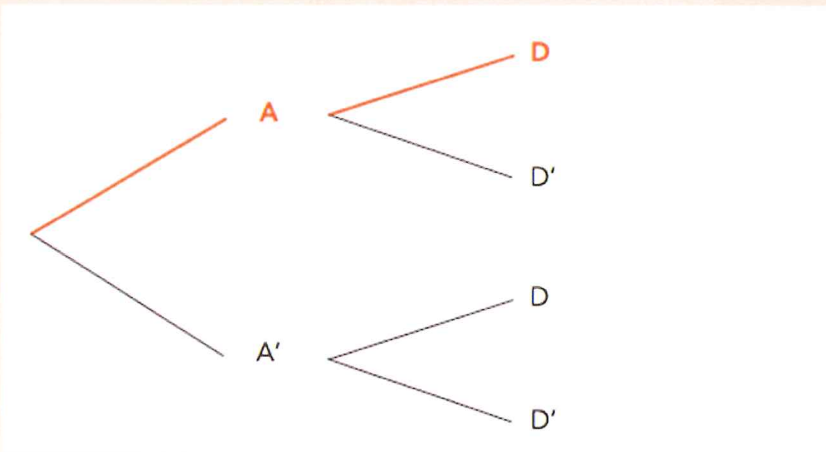
Those who take Art and Design.

All of those who take Art.

$$P(D/A) = \frac{\text{Those who take Design and Art}}{\text{Those who take just Art}}$$

$$= \frac{0.08\dot{3}}{0.15} = 0.\dot{5}$$

Alternative method: Use a probability tree.



$$P(D/A) = \frac{0.08\dot{3}}{0.15}$$

$$= 0.\dot{5}$$

Independent Events

Independent events are when the occurrence on one has no effect on the other.

$P(A \cap B) = P(A) \times P(B)$

If this is true the two events are independent

$P(A \cap B) \neq P(A) \times P(B)$

If this is the case then the two events are NOT independent

Example:

P(having a birthday in December) P(becoming a prefect)

We would expect these two events to be independent.

In General:

So events are independent if: $P(A) = P(A/B)$ or $P(A \cap B) = P(A) \times P(B)$

Knowing that B has occurred makes no difference to the probability of A occurring.

If $P(A \cap B) > P(A) \times P(B)$ then there is a positive association between events A and B ie The occurrence of 'B' increases the chance of event 'A' (and vice versa – whatever seems logical)

If $P(A \cap B) < P(A) \times P(B)$ then there is a negative association between events A and B ie The occurrence of 'B' decreases the chance of event 'A' (and vice versa)

Exercises on Probability Tables

1. $P(A) = 0.4$, $P(B) = 0.5$ $P(A \cap B) = 0.1$

	A	A'	
B	0.1	0.4	0.5
B'	0.3	0.2	0.5
	0.4	0.6	1

Complete the table and answer the following questions.

(a) What is the probability that A or B occurs? $0.4 + 0.4 + 0.3 = 0.8$

(b) What is the probability that neither A nor B occurs? 0.2

(c) Are A and B independent? $P(A) \times P(B) = 0.4 \times 0.5$
$= 0.2 \neq P(A \cap B) = 0.1$ not independent

(d) Find $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.5} = 0.2$

(e) Are A and B mutually exclusive?

No as $P(A \cap B) = 0.1$ not 0

2. P(A) is 0.4, P(B) is 0.6, and A and B are independent events.

as $P(A) \times P(B) = P(A \cap B)$

	A	A'	
B	0.24	0.36	0.6
B'	0.16	0.24	0.4
	0.4	0.6	1

Complete the table and answer the following questions.

(a) What is the probability that A or B occurs? $0.24 + 0.16 + 0.36 = 0.76$

(b) What is $P(B|A')$? $= \frac{P(B \cap A')}{P(A')} = \frac{0.36}{0.6} = 0.6$

(c) Are A and B mutually exclusive?

No as $P(A \cap B) = 0.24$ not 0

Both can happen

5. $P(A \cup B) = 0.7$, $P(A) = 0.5$, $P(B) = 0.4$

	A	A'	
B	0.2	0.2	0.4
B'	0.3	0.3	0.6
	0.5	0.5	1

$1 - 0.7$

Complete the table and answer the following questions.

(a) What is the probability that both of A and B occur? 0.2

(b) What is the probability that neither A nor B occurs? 0.3

(c) What is $P(A|B)$? $\frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4} = 0.5$

(d) Are A and B independent events?

$P(A) \times P(B) = 0.5 \times 0.4 = 0.2$ $\quad P(A \cap B) = 0.2$ So yes they are independent

6. $P(A) = 0.4$, $P(B) = 0.5$, $P(A|B) = 0.5$

$\frac{P(A \cap B)}{0.5} = 0.5$

	A	A'	
B	0.25	0.25	0.5
B'	0.15	0.35	0.5
	0.4	0.6	1

Complete the table and answer the following questions.

(a) What is the probability that A or B occurs? $0.25 + 0.25 + 0.15 = 0.65$

(b) What is the probability that neither A nor B occurs? 0.35

(c) What is $P(B|A)$? $\frac{P(A \cap B)}{P(A)} = \frac{0.25}{0.4} = 0.625$

(d) Are A and B independent?

$P(A) \times P(B) = 0.4 \times 0.5$
$= 0.2$

$P(A \cap B) = 0.25$

These two values are not equal so not independent

Relative Risk

Relative Risk is used to compare the risk for two groups. It gives a measure of the impact of the behaviour or treatment group is exposed to.

Usually the baseline group (denominator) is the non-treatment group (this will be in the wording of the question).

$$\text{Relative Risk} = \frac{\text{Risk for treatment group}}{\text{Risk for non treatment group}}$$

Relative risk does not have to be between 0 and 1

RR of 2.5 means = you are 2.5 times more likely to... (or a 150% increase)

RR of 1 means = you are equally likely to......

RR of 0.8 means = there is 0.8 times the chance of.... (a decreased risk eg exercise on the chance of a heart attack) (This is 20% LESS risk for those that exercise)

or 0.8 times as likely.....

Group A	Heart Disease	None	Total
Overweight	24	142	166
Not Overwt	14	170	184
Total	38	312	350

What is the Relative Risk of heart disease in an overweight male compared with a non-overweight male.

$$\frac{24/166}{14/184} = 1.9$$

What can we conclude?

An overweight male is 1.9 times as likely to get heart disease.

Example: A trial was done on 1000 sheep in order to determine the effectiveness of a vaccine to protect the sheep against a disease. The vaccine was given to 600 sheep, and the remaining 400 were not vaccinated. Records were kept of how many sheep got the disease.

	Disease	No disease	Totals
Vaccinated	39	561	**600**
Unvaccinated	87	313	**400**
Totals	**126**	**874**	**1000**

a Calculate the absolute risk that a sheep got the disease.

$$\text{Absolute risk} = \frac{\text{Total number that have disease}}{\text{Total number of sheep}} = \frac{126}{1000} = 0.126$$

b Calculate the absolute risk that an unvaccinated sheep got the disease.

$$\text{Absolute risk} = \frac{\text{Number of unvaccinated sheep that got disease}}{\text{Number of unvaccinated sheep}} = \frac{87}{400} = 0.2175$$

c Calculate the absolute risk that a vaccinated sheep got the disease.

$$\text{Absolute risk} = \frac{\text{Number of vaccinated sheep that got disease}}{\text{Number of vaccinated sheep}} = \frac{39}{600} = 0.065$$

d Calculate the relative risk that an unvaccinated sheep gets the disease, compared with a vaccinated sheep.

$$\text{Relative risk} = \frac{\text{Probability of getting disease if unvaccinated}}{\text{Probability of getting disease if vaccinated}} = \frac{0.2175}{0.065} = 3.346$$

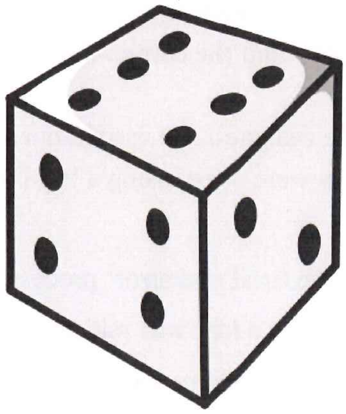
e Explain what your result means.

An unvaccinated sheep is more than three times more likely to get the disease than a vaccinated sheep.

Theoretical vs Experimental vs Actual Probability

Theoretical Probability

Theoretical Probability also known as model probability is when we use a tool to find out the theoretical probability of an event occurring



Example:

The theoretical probability of rolling a 3 on a dice is $\frac{1}{6}$ or 0.16666 or 16%

Experimental Probability

Experimental probability is based on the number of times the event occurs out of the total number of trials.

Example:

Sam rolled a dice 50 times. A 3 appeared 10 times.

Then the experimental probability of rolling a 3 is 10 out of 50 or 20%.

In probability an experiment is one or more trials of a probability situation.

Actual Probability

Also known as true probability is the (almost always) unknown actual probability that an event will occur in a given situation.

Example:

The actual probability of a coin landing heads up is affected by the position from which it is tossed, the asymmetry of the two faces of the coin etc, so is not exactly 0.5, though the probability of a fair coin landing heads will be very close to 0.5.

(a) Calculate the theoretical probability of a person using a 'trial and error' process taking more than two key attempts before both locks are open. Compare this probability with the results from Sene's simulation and discuss any differences.

P(2 key attempts) = $\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$

P(more than 2) = $1 - \frac{1}{30}$

$= \frac{29}{30} = 0.9667$

(b) Sene used the central 90% of her simulation results to check if the number of key attempts Emma took (four) was likely if the 'trial and error' process was used, and concluded it was. Discuss if the results of Sene's simulation support this conclusion.

90% between 3 - 10 attempts

4 is within this range ∴ simulation

results seem appropriate

(c) Calculate the theoretical probability that a person using Emma's process takes four key attempts before both locks are open.

✓ ✗ ✗ ✓ $\frac{1}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{30}$

✗ ✓ ✗ ✓ $\frac{5}{6} \times \frac{1}{5} \times \frac{4}{5} \times \frac{1}{4} = \frac{1}{30}$

✗ ✗ ✓ ✓ $\frac{5}{6} \times \frac{4}{5} \times \frac{1}{4} \times \frac{1}{5} = \frac{1}{30}$

$\frac{1}{30} + \frac{1}{30} + \frac{1}{30} = \frac{3}{30}$ or $\frac{1}{10}$